Rational Approximation, Hardy Space - Decomposition of Functions in $L_p, p < 1$: Further Results in Relation to Fourier Spectrum Characterization of Hardy Spaces

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Abstract

Subsequent to our recent work on Fourier spectrum characterization of Hardy spaces $H^p(\mathbb{R})$ for the index range $1 \le p \le \infty$, in this paper we prove further results on rational Approximation, integral representation and Fourier spectrum characterization of functions in the Hardy spaces $H^p(\mathbb{R}), 0 < p \le \infty$, with particular interest in the index range $0 < p \le 1$. We show that the set of rational functions in $H^p(\mathbb{C}_{+1})$ with the single pole $-i$ is dense in $H^p(\mathbb{C}_{+1})$ for $0 < p < \infty$. Secondly, for $0 < p < 1$, through rational function approximation we show that any function f in $L^p(\mathbb{R})$ can be decomposed into a sum $g + h$, where g and h are, in the $L^p(\mathbb{R})$ convergence sense, the non-tangential boundary limits of functions in, respectively, $H^p(\mathbb{C}_{+1})$ and $H^p(\mathbb{C}_{-1})$, where $H^p(\mathbb{C}_k)$ $(k = \pm 1)$ are the Hardy spaces in the half plane $\mathbb{C}_k = \{z = x + iy : ky > 0\}$. We give Laplace integral representation formulas for functions in the Hardy spaces H^p, $0 < p \le 2$. Besides one in the integral representation formula we give an alternative version of Fourier spectrum characterization for functions in the boundary Hardy spaces H^p for $0 < p \le 1$.

Key Words The Paley-Wiener Theorem, Hardy Space

1 Introduction

The classical Hardy space $H^p(\mathbb{C}_k)$, $0 < p < +\infty, k = \pm 1$, consists of the functions f analytic in the half plane $\mathbb{C}_k = \{z = x + iy : ky > 0\}$. They are

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Banach spaces for $1 \leq p < \infty$ under the norms

$$\|f\|_{H^p_k} = \sup_{ky>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}};$$

and complete metric spaces for $0 < p < 1$ under the metric functions

$$d(f,g) = \sup_{ky>0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx.$$

A function $f \in H^p(\mathbb{C}_k)$ has non-tangential boundary limits (*NTBLs*) $f(x)$ for almost all $x \in \mathbb{R}$. The corresponding boundary function belongs to $L^p(\mathbb{R})$. For $1 \leq p < \infty$,

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}} = \|f\|_{H^p_k}.$$

For $p = \infty$ the Hardy spaces $H^\infty(\mathbb{C}_k)$ $(k = \pm 1)$ are defined to be the set of bounded analytic functions in $\mathbb{C}_k$. They are Banach spaces under the norms

$$\|f\|_{H^\infty_k} = \sup\{|f(z)| \ : \ z \in \mathbb{C}_k\}.$$

As for the finite indices p cases any $f \in H^\infty(\mathbb{C}_k)$ has non-tangential boundary limit (NTBL) $f(x)$ for almost all $x \in \mathbb{R}$. Similarly, we have

$$\|f\|_\infty = \text{ess sup}\{|f(x)| \ : \ x \in \mathbb{R}\} = \|f\|_{H^\infty(\mathbb{C}_k)}.$$

We note that $g(z) \in H^p(\mathbb{C}_{-1})$ if and only if the function $f(z) = \overline{g(\overline{z})} \in H^p(\mathbb{C}_{+1})$. The correspondence between their non-tangential boundary limits and the functions themselves in the Hardy spaces is an isometric isomorphism. We denote by $H^p_k(\mathbb{R})$ the spaces of the non-tangential boundary limits, or, precisely,

$$H^p_k(\mathbb{R}) = \left\{ f \ : \mathbb{R} \to \mathbb{C}, f \text{ is the NTBL of a function in } \ H^p(\mathbb{C}_k) \right\}.$$

For $p = 2$ the Boundary Hardy spaces $H^2_k(\mathbb{R})$ are Hilbert spaces.

We will need some very smooth classes of analytic functions that are dense in $H^p(\mathbb{C}_{+1})$ and will play the role of the polynomials in the disc case. J,B. Garnett in [5] shows the following results.

Theorem A ([5]) Let N be a positive integer. For $0 < p < \infty$, $pN > 1$, the class $\mathfrak{A}_N$ is dense in $H^p(\mathbb{C}_{+1})$, where $\mathfrak{A}_N$ is the family of $H^p(\mathbb{C}_{+1})$ functions satisfying

(i) $f(z)$ is infinitely differentiable in $\overline{\mathbb{C}_{+1}}$,

(ii) $|z|^N f(z) \to 0$ as $z \to\to \infty, z \in \overline{\mathbb{C}_{+1}}$.

We shall notice that the condition $pN > 1$ implies the class $\mathfrak{A}_N$ is contained in $H^p(\mathbb{C}_{+1})$. Let α be a complex number and $\mathfrak{R}_N(\alpha)$ the family of rational

functions $f(z) = (z+\alpha)^{-N-1}P((z+\alpha)^{-1})$, $P(w)$ are polynomials. We notice that the class $\mathfrak{R}_N(\alpha)$ is contained in the class $\mathfrak{A}_N$ for $\mathrm{Im}\alpha > 0$.

The tasks of this paper are three-fold. The first, replacing the class $\mathfrak{A}_N$ by the class $\mathfrak{R}_N(i)$, we will generalize Theorem A as

Theorem 1 Let N be a positive integer. For $0 < p < \infty,\ Np > 1$, the class $\mathfrak{R}_N(i)$ is dense in $H^p(\mathbb{C}_{+1})$.

Corollary 1 Let N be a positive integer. For $0 < p < \infty,\ Np > 1$, the class $\mathfrak{R}_N(-i)$ is dense in $H^p(\mathbb{C}_{-1})$.

The second task is decomposition of functions in $L^p(\mathbb{R}), 0 < p < 1$, into sums of the corresponding Hardy space functions in $H^p_{+1}(\mathbb{R})$ and in $H^p_{-1}(\mathbb{R})$ through rational functions approximation, and, in fact, by using what we call as rational atoms.

Theorem 2 (Hardy Spaces Decomposition of L^p Functions For $0 < p < 1$) Suppose that $0 < p < 1$ and $f \in L^p(\mathbb{R})$. Then, there exist a positive constant A_p and two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_{+1})$, $Q_k \in H^p(\mathbb{C}_{-1})$ and

$$\sum_{k=1}^{\infty}\left(\|P_k\|^p_{H^p_{+1}} + \|Q_k\|^p_{H^p_{-1}}\right) \leq A_p\|f\|^p_p, \tag{1}$$

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n}(P_k + Q_k)||_p = 0. \tag{2}$$

Moreover,

$$g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^p(\mathbb{C}_{+1}), \quad h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^p(\mathbb{C}_{-1}), \tag{3}$$

and $g(x)$ and $h(x)$ are the non-tangential boundary values of functions for $g \in H^p(\mathbb{C}_{+1})$ and $h \in H^p(\mathbb{C}_{-1})$, respectively, $f(x) = g(x)+h(x)$ almost everywhere, and

$$\|f\|^p_p \leq \|g\|^p_p + \|h\|^p_p \leq A_p\|f\|^p_p,$$

that is, in the sense of $L^p(\mathbb{R})$,

$$L^p(\mathbb{R}) = H^p_{+1}(\mathbb{R}) + H^p_{-1}(\mathbb{R}).$$

For the uniqueness of the decomposition, we can ask the following question: what is the intersection space $H^p_{+1}(\mathbb{R}) \bigcap H^p_{-1}(\mathbb{R})$? A.B. Aleksandrov ([1] and [2]) gives an answer for this problem.

Theorem B ([1] and [2]) Let $0 < p < 1$ and X^p denote the L^p closure of the set of $f \in L^p(\mathbb{R})$ which can be written in the form

$$f(x) = \sum_{j=1}^{N} \frac{c_j}{x - a_j}, \quad a_j \in \mathbb{R},\ c_j \in \mathbb{C}.$$

Then

$$X^p = H^p_{+1}(\mathbb{R}) \bigcap H^p_{-1}(\mathbb{R}).$$

A.B. Aleksandrov's proof ([1] and [2]) is rather long involving vanishing moments and the Hilbert transformation. We present a more straightforward proof for this result.

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined, for $x \in \mathbb{R}$, by

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ixt} dt.$$

Based on the Fourier transformation defined for $L^1(\mathbb{R})$-functions, Fourier transformation can be extended to $L^2(\mathbb{R})$, and then to $L^p(\mathbb{R}), 0 < p < 2$, and finally to $L^p(\mathbb{R}), 2 < p \leq \infty$, the latest bring in the distribution sense.

The classical Paley-Wiener Theorem deals with the Hardy $H^2(\mathbb{C}_{+1})$ space ([3],[4], [5],[6] and [11]) asserting that $f \in L^2(\mathbb{R})$ is the NTBL of a function in $H^2(\mathbb{C}_{+1})$ if and only if $\text{supp}\hat{f} \subset [0, \infty)$. Moreover, in such case, the integral representation

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} \hat{f}(t) dt \tag{4}$$

holds.

We recall that Fourier transform of a tempered distribution T is defined through the relation

$$(\widehat{T}, \varphi) = (T, \hat{\varphi})$$

for φ in the Schwarz class $\mathbb{S}$. This coincides with the traditional definition of Fourier transformation for functions in $L^p(\mathbb{R}), 1 \leq p \leq 2$. A measurable function f satisfying

$$\frac{f(x)}{(1+x^2)^m} \in L^p(\mathbb{R}) \quad (1 \leq p \leq \infty)$$

for some positive integer integer m is called a tempered L^p function (when $p = \infty$ such a function is often called a slowly increasing function). The Fourier transform is a one to one mapping from $\mathbb{S}$ onto $\mathbb{S}$.

It is proved in [8] that a function in $H^p_{+1}(\mathbb{R}), 1 \leq p \leq \infty$, induces a tempered distribution T_f such that $\text{supp}\hat{T}_f \subset [0, \infty)$. In [9], the converse of the result is proved: Let T_f be the tempered distribution induced by f in $L^p(\mathbb{R}), 1 \leq p \leq \infty$. If $\text{supp}\hat{T}_f \subset [0, \infty)$, then $f \in H^p_{+1}(\mathbb{R})$.

The third task of this paper is to extend the above mentioned Fourier spectrum results, as well as the formula (4) to $0 < p < 1$.

Theorem 3 (Integral Representation Formula For Index Range $0 < p \leq 1$) If $0 < p \leq 1,\ f \in H^p(\mathbb{C}_{+1})$, then there exist a positive constant A_p, depending only on p, and a slowly increasing continuous function F whose support is contained in $[0, \infty)$, satisfying that, for φ in the Schwarz class $\mathbb{S}$,

$$(F, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}} f(x+iy) \hat{\varphi}(x) dx,$$

and that

$$|F(t)| \leqslant A_p \|f\|_{H^p_{+1}} |t|^{\frac{1}{p}-1}, \quad (t \in \mathbb{R}) \tag{5}$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty F(t) e^{itz} dt \qquad (z \in \mathbb{C}_{+1}). \tag{6}$$

P. Duren cites on page 197 of [4] that the argument to prove the integral representation (4) for $p = 2$ can be generalized to give an analogous representation for $H^p(\mathbb{C}_{+1})$-functions for $1 \le p < 2$. A proof for the range $1 \le p < 2$, in fact, is not obvious, and so far has not appeared in the literature, as far as concerned by the authors. We are to prove the following theorem corresponding to what Duren stated.

Theorem C ([4], integral Representation Formula For Index Range $1 \le p \le 2$) Suppose $1 \le p \le 2$, $f \in L^p(\mathbb{R})$. Then $f \in H^p_{+1}(\mathbb{R})$ if and only if $\mathrm{supp}\hat{f} \subset [0, +\infty)$. Moreover, under such conditions the integral representation (4) holds.

We, in fact, prove analogous formulas for all the cases $0 < p \le 2$. For the range $0 < p < 1$ we need to prove extra estimates to guarantee the integrability (See the proof of Theorem 3). The idea of using rational approximation is motivated by the studies of Takenaka-Malmquist systems in Hardy H^p spaces for $1 \le p \le \infty$ ([12], [10]). For the range of $1 \le p \le \infty$ this aspect is related to the Plemelj formula in terms of Hilbert transform that has immediate implication to Fourier spectrum characterization in the case. For the range of $0 < p < 1$ the Plemelj formula approach is not available.

2 Proofs of theorems

We need the following Lemmas.

Lemma 1 Suppose that $0 < p < 1$ and R is a rational function with $R \in L^p(\mathbb{R})$. For $k = \pm 1$, if $R(z)$ is analytic in the half plane $\mathbb{C}_k$, then $R \in H^p(\mathbb{C}_k)$.

Proof Let $0 < p < 1$, $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are co-prime polynomials with degrees m and n, respectively. Then there exists a constant $c \neq 0$ such that

$$\lim_{z \to \infty} R(z) z^{n-m} = c.$$

As consequence, there exists a constant $M_0 > 1$ such that

$$\frac{|c|}{2} |z|^{m-n} \le |R(z)| \le 2|c||z|^{m-n}, \quad |z| > M_0.$$

$R \in L^p([M_0, \infty))$ implies that $p(m-n) < -1$, and so for $y \ge 0$,

$$\int_{|x|>M_0} |R(x+iy)|^p dx \leqslant (2|c|)^p \int_{|x|>M_0} |x+iy|^{p(m-n)} dx$$

$$\leqslant (2|c|)^p \int_{|x|>M_0} |x|^{p(m-n)} dx \leqslant \frac{2^{p+1}|c|^p}{p(n-m)-1} < \infty.$$

Similarly for $y > M_0$,

$$\int_{|x|\leq M_0} |R(x+iy)|^p dx \leqslant (2|c|)^p \int_{|x|\leq M_0} |x+iy|^{p(m-n)} dx$$

$$\leqslant 2(2|c|)^p M_0^{p(m-n)+1} < \infty.$$

If $R(z)$ is analytic in the upper half plane $\mathbb{C}_{+1}$, then $Q(z) \neq 0$ for $z \in \mathbb{C}_{+1}$. If, furthermore, $Q(x) \neq 0$ for $x \in \mathbb{R}$, then $R(z)$ is continuous in the rectangle $E_0 = [-M_0, M_0] \times [0, M_0]$, and so $R \in H^p(\mathbb{C}_{+1})$. Otherwise, the null set $N_Q = \{a \in \mathbb{R} : Q(a) = 0\}$ of Q in $\mathbb{R}$ is a finite set. Let $N_Q = \{a_1, a_2, \cdots, a_q\}$ with $a_1 < a_2 < \cdots < a_q$, and $P(a_k) \neq 0(\ k = 1, 2, \cdots, q)$. Then there exists a polynomial $Q_1(z)$ with $Q_1(a_k) \neq 0$ ($k = 1, 2, \cdots, q$) and positive integers $l_k(\ k = 1, 2, \cdots, q)$ such that

$$Q(z) = (z-a_1)^{l_1}(z-a_2)^{l_2} \cdots (z-a_q)^{l_q} Q_1(z);$$

and, there exist positive constants δ, ε_0 and $M_1 > \varepsilon_0$ such that

$$\varepsilon_0 \leq |R(z)(z-a_k)^{l_k}| \leq M_1,$$

for $z = x+iy \in I_k = \{z = x+iy : 0 < |x-a_k| \leq \delta, \ 0 \leq y \leq \delta\}$.

Therefore,

$$\int_{|x-a_k|\leq\delta} |R(x)|^p dx \geq \varepsilon_0^p \int_{|x-a_k|\leq\delta} |x-a_k|^{-pl_k} dx.$$

The fact that $R \in L^p([a_k-\delta, a_k+\delta])$ implies that $pl_k < 1$. So, for $y \in [0, \delta]$,

$$\int_{|x-a_k|\leq\delta} |R(x+iy)|^p dx \leqslant M_1^p \int_{|x-a_k|\leq\delta} |x+iy-a_k|^{-pl_k} dx$$

$$\leqslant M_1^p \int_{|x-a_k|\leq\delta} |x-a_k|^{-pl_k} dx = \frac{2M_1^p \delta^{1-pl_k}}{1-pl_k} < \infty.$$

Since the poles of $R(z)$ in the closed upper half plane are identical with N_Q, $R(z)$ is continuous in the bounded closed set

$$\{z \in E_0 : z \notin I_k, \ k = 1, 2, \cdots, q\}.$$

Therefore

$$\int_{|x|\leq M_0} |R(x+iy)|^p dx$$

is uniformly bounded for $y \in [0, M_0]$. This proves that $R \in H^p(\mathbb{C}_{+1})$. If $R(z)$ is analytic in the lower half plane $\mathbb{C}_{-1}$, Lemma 1 can be proved similarly.

Lemma 2 If $0 < p \leq 1$, $f \in L^p(\mathbb{R})$, then, for $\varepsilon > 0$, there exists a sequence of rational functions $\{R_k(z)\}$, whose poles are either i or $-i$, such that

$$\sum_{k=1}^{\infty} ||R_k||_p^p \leq (1+\varepsilon)\|f\|_p^p \tag{7}$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n} R_k||_p = 0. \tag{8}$$

Proof For the case $0 < p < 1$, we can assume that $\|f\|_p^p > 0$. The fractional linear mapping (the Cayley Transformation)

$$z = \alpha(w) = i\frac{1-w}{1+w}$$

is a conformal mapping from the unit disc $U = \{w : |w| < 1\}$ to the upper half plane $\mathbb{C}_{+1}$, its inverse mapping is

$$\beta(z) = \frac{i-z}{z+i}.$$

Let $x = \alpha(e^{i\theta}),\ \theta \in [-\pi, \pi]$. Then $x = \tan\frac{\theta}{2}$ and $dx = \frac{d\theta}{1+\cos\theta}$. So,

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \int_{-\pi}^{\pi} \left| f(\tan\frac{\theta}{2}) \right|^p \frac{d\theta}{1+\cos\theta} < \infty.$$

This implies that the function

$$g(\theta) = \frac{f(\tan\frac{\theta}{2})}{(1+\cos\theta)^{\frac{1}{p}}} \in L^p([-\pi, \pi]).$$

Since the set of trigonometric polynomials is dense in $L^p([-\pi, \pi])$, there exists a sequence of rational functions $\{r_k(w)\}$, whose poles can only be zero, with the expression $r_k(e^{i\theta}) = \sum_{j=-m_k}^{m_k} c_{k,j} e^{ij\theta}$, such that

$$\lim_{k\to\infty} ||g(\theta) - r_k(e^{i\theta})||_{L^p([-\pi,\pi])} = 0.$$

Furthermore, for any $\varepsilon > 0$, the sequence of rational functions $\{r_k(w)\}$ can be chosen so that

$$||g(\theta) - r_k(e^{i\theta})||^p_{L^p([-\pi,\pi])} \leq \frac{A_\varepsilon}{4^{k+3}},$$

where $A_\varepsilon = \|f\|_p^p \varepsilon$. Since $0 < p < 1$, there exists a positive integer l_p such that $1 < p2^{l_p} \leq 2$. Take $m = 2^{l_p - 1}$. Then m is a positive integer satisfying $1 < 2pm \leq 2$. Thus we have $0 \geq 2(pm-1) > -1$, and, as consequence, the function

$$g_1(\theta) = (2\sin^2\theta)^{pm-1} \in L^1\left[0, \frac{\pi}{2}\right].$$

The function $g_2(x) = x^{\frac{1}{p}-m}$ is also continuous in the interval $[0, 2]$. The Weierstrass Theorem asserts that there exists a sequence of polynomials $\{q_k(x)\}$ such that

$$|g_2(x) - q_k(x)| < \frac{A_\varepsilon}{M_k^p C_1 4^{k+3}}, \tag{9}$$

where

$$M_k = \sum_{j=-m_k}^{m_k} |c_{k,j}| + 1, \quad C_1 = \int_0^{\frac{\pi}{2}} g_1(\theta) d\theta.$$

Thus we obtain

$$\int_0^{\frac{\pi}{2}} |(2\sin^2\theta)^{\frac{1}{p}-m} - q_k(2\sin^2\theta)|^p g_1(\theta) d\theta \le \frac{A_\varepsilon}{M_k 4^{k+3}}.$$

The function

$$s_k(e^{i\theta}) = r_k(e^{i\theta}) q_k(1+\cos\theta)(1+\cos\theta)^m$$

is a trigonometric polynomial, and satisfies

$$J_k = \int_{-\pi}^{\pi} \left| r_k(e^{i\theta}) - \frac{s_k(e^{i\theta})}{(1+\cos\theta)^{\frac{1}{p}}} \right|^p d\theta.$$

$$\le M_k^p \int_{-\pi}^{\pi} |1 - q_k(1+\cos\theta)(1+\cos\theta)^{m-\frac{1}{p}}|^p d\theta$$

$$= M_k^p \int_{-\pi}^{\pi} |(1+\cos\theta)^{\frac{1}{p}-m} - q_k(1+\cos\theta)|^p (1+\cos\theta)^{pm-1} d\theta$$

$$= M_k^p \int_{-\pi}^{\pi} |g_2(1+\cos\theta) - q_k(1+\cos\theta)|^p (1+\cos\theta)^{pm-1} d\theta.$$

Hence, by (9),

$$J_k \le \frac{A_\varepsilon}{C_1 4^{k+3}} \int_{-\pi}^{\pi} (1+\cos\theta)^{pm-1} d\theta = \frac{A_\varepsilon}{C_1 4^{k+2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{pm-1} d\theta$$

$$\le \frac{A_\varepsilon}{C_1 4^{k+2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin\theta)^{pm-1} d\theta = \frac{2A_\varepsilon}{4^{k+2}}.$$

Finally, the function

$$g_k(\theta) = \frac{s_k(e^{i\theta})}{(1+\cos\theta)^{\frac{1}{p}}}$$

satisfies

$$||g - g_k||^p_{L^p([-\pi,\pi])}$$

$$\le ||g - r_k(e^{i\cdot})||^p_{L^p([-\pi,\pi])} + ||r_k(e^{i\cdot}) - g_k||^p_{L^p([-\pi,\pi])} \le \frac{A_\varepsilon}{4^{k+1}},$$

and

$$||g-g_k||^p_{L^p([-\pi,\pi])} = \int_{-\pi}^{\pi}\left|f(\tan\frac{\theta}{2})-s_k(e^{i\theta})\right|^p\frac{d\theta}{1+\cos\theta}$$
$$= \int_{-\infty}^{\infty}\left|f(x)-s_k\left(\frac{i-x}{x+i}\right)\right|^p dx \le \frac{A_\varepsilon}{4^{k+1}}.$$

The function

$$Q_k(z) = s_k\left(\frac{i-z}{z+i}\right)$$

is a rational function whose poles are either i or $-i$, and

$$||Q_k||^p_p = \int_{-\infty}^{\infty}|Q_k(x)|^p dx = \int_{-\infty}^{\infty}\left|s_k\left(\frac{i-x}{x+i}\right)\right|^p dx \le \|f\|^p_p + \frac{A_\varepsilon}{4^{k+1}}$$

and

$$||f-Q_k||^p_p = \int_{-\infty}^{\infty}\left|f(x)-s_k\left(\frac{i-x}{x+i}\right)\right|^p dx \le \frac{2A_\varepsilon}{4^{k+1}}.$$

Therefore, the sequence of rational functions $\{Q_k(z)\}$ can be chosen so that

$$||Q_k - Q_{k-1}||^p_p \le \frac{A_\varepsilon}{4^k}. \quad (k=2,3,\cdots)$$

Let

$$R_1(z) = Q_1(z),\ R_k(z) = Q_k(z) - Q_{k-1}(z), \quad (k=2,3,\cdots).$$

$\{R_k(z)\}$ is a sequence of rational functions whose poles can only be i or $-i$, satisfying (7) and (8). This completes the proof of Lemma 2.

Lemma 3 Suppose that $0<p<1$ and that $R\in L^p(\mathbb{R})$ is a rational function whose poles are contained in $\{i,-i\}$, then there exist two rational functions $P\in H^p(\mathbb{C}_{+1})$ and $Q\in H^p(\mathbb{C}_{-1})$ such that $R(z)=P(z)+Q(z)$ and

$$\|P\|^p_{H^p_{+1}} + \|Q\|^p_{H^p_{-1}} \le \left(1+\frac{4\pi}{1-p}\right)\|R\|^p_p,$$

Proof Let $0<p<1$, $R\in L^p(\mathbb{R})$, and R be a rational function whose poles are contained in $\{i,-i\}$. Then $R(z)$ can be written as

$$R(z) = \sum_{k=-n}^{n} c_k(\beta(z))^k, \quad \text{where} \quad \beta(z) = \frac{i-z}{z+i}.$$

Therefore, $\beta(x) = e^{i\theta(x)}$, where $\theta(x) = \arg(i-x) - \arg(x+i) \in (-\pi,\pi)$ for $x\in\mathbb{R}$. Define, for each $\varphi\in\mathbb{R}$,

$$P(z,\varphi) = \frac{(\beta(z))^m R(z)}{(\beta(z))^m - e^{i\varphi}}, \quad Q(z,\varphi) = \frac{(\beta(z))^{-m} R(z)}{(\beta(z))^{-m} - e^{-i\varphi}},$$

where m is any positive integer greater than the positive integer n. By Fubini's theorem,

$$I = \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} |P(x,\varphi)|^p dx d\varphi = \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} \frac{|\beta(x)|^{mp}|R(x)|^p}{|(\beta(x))^m - e^{i\varphi}|^p} dx d\varphi$$

$$= \int_{-\infty}^{+\infty}\int_{-\pi}^{\pi} \frac{|R(x)|^p}{|1-e^{i(\varphi - m\theta(x))}|^p} d\varphi dx.$$

Observing that

$$\int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1-e^{i\varphi - im\theta(x)}|^p} = \int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1-e^{i\varphi}|^p} = \int_{-\pi}^{\pi} \frac{d\varphi}{\sin^p \frac{\varphi}{2}} \le 4\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\frac{2}{\pi}\varphi)^p} \le \frac{2\pi}{1-p},$$

we obtain that

$$I \le \frac{2^{1-p}\pi}{1-p}\int_{-\infty}^{+\infty} |R(x)|^p dx.$$

Therefore, there is a real number φ such that

$$\int_{-\infty}^{+\infty} |P(x,\varphi)|^p dx \le \frac{2\pi}{1-p}\int_{-\infty}^{+\infty} |R(x)|^p dx.$$

For this specially chosen real number φ, by defining $P(z) = P(z,\varphi)$, $Q(z) = Q(z,\varphi)$, we have $R(z) = P(z) + Q(z)$. Since $m > n$, the functions P and Q are rational functions and the poles of $P(z)$ and $Q(z)$ both are contained in the set $\{i\}\bigcup\{x_k : k = 0,1,2,\cdots,n-1\}$, where through the Cayley Transformation

$$x_k = \alpha(e^{\frac{i}{n}(\varphi+2k\pi)}) = \tan^2(\frac{1}{2n}(\varphi+2k\pi))$$

are real numbers. Therefore, $P(z)$ is analytic in the upper half plane $\mathbb{C}_{+1}$, and $Q(z)$ is analytic in the lower half plane $\mathbb{C}_{-1}$, and

$$\int_{-\infty}^{+\infty} |P(x)|^p dx \le \frac{2\pi}{1-p}\int_{-\infty}^{+\infty} |R(x)|^p dx$$

$$\int_{-\infty}^{+\infty} |Q(x)|^p dx \le \left(1 + \frac{2\pi}{1-p}\right)\int_{-\infty}^{+\infty} |R(x)|^p dx.$$

By Lemma 1, $P \in H^p(\mathbb{C}_{+1})$, $Q \in H^p(\mathbb{C}_{-1})$. This completes the proof of Lemma 3.

Proof of Theorem 1 If $f \in H^p(\mathbb{C}_{+1})$, $Np > 1$, then, for any $\varepsilon > 0$, by Theorem A, there exists function f_N in $H^p(\mathbb{C}_{+1})\bigcap C^\infty(\overline{\mathbb{C}_{+1}})$ such that

$$\lim_{|z|\to 0, Imz\ge 0} |z|^{N+1}|f_N(z) = 0$$

and

$$\|f_N - f\|_{H^p_{+1}} < \varepsilon.$$

The fractional linear mapping (the Cayley Transformation)

$$z = \alpha(w) = i\frac{1-w}{1+w}$$

is a conformal mapping from the unit disc $U = \{w : |w| < 1\}$ to the upper half plane $\mathbb{C}_+$, its inverse mapping is

$$w = \beta(z) = \frac{i-z}{z+i}.$$

Let $h_N(w) = f_N(\alpha(w))$ and $h_N(-1) = 0$, then $h_N(w)$ is continuous in the closed disc $\overline{U}$ and

$$h_N(w)\left(i\frac{1-w}{1+w}\right)^{N+1} \to 0, \quad w \in \overline{U} \setminus \{-1\}, w \to -1.$$

So,

$$\frac{h_N(w)}{(1+w)^{N+1}} \to 0, \quad w \to -1,\ |w| \le 1,\ w \ne 1.$$

If let $\tilde{h}_N(w) = \frac{h_N(w)}{(1+w)^{N+1}}$ and $\tilde{h}_N(-1) = 0$, then $\tilde{h}_N(w)$ is analytic in the unit disc U and continuous in the closed unit disc $\overline{U}$. Therefore, there exists polynomial P_N such that

$$\left|\frac{h_N(w)}{(1+w)^{N+1}} - P_N(1+w)\right| < \varepsilon, \quad |w| \le 1, w \ne -1.$$

Thus,

$$|f_N(\alpha(w)) - (1+w)^{N+1}P_N(1+w)| < \varepsilon|1+w|^{N+1}, |w| \le 1, w \ne -1.$$

Since $z = \alpha(w)$ and $w = \frac{i-z}{i+z}$, the above inequality become

$$\left|f_N(z) - \left(\frac{2i}{i+z}\right)^{N+1} P_N\left(\frac{2i}{i+z}\right)\right| < \varepsilon\left|\frac{2i}{i+z}\right|^{N+1},\ Imz \ge 0.$$

Therefore, we obtain

$$\int_{-\infty}^{\infty} |f_N(x+iy) - R(x+iy)|^p\ dx \le \varepsilon^p 2^{(N+1)p} \int_{-\infty}^{\infty} \left|\frac{1}{x^2+1}\right|^{(N+1)p} dx,$$

where $R(z) = (\frac{2i}{i+z})^{N+1} P_N(\frac{2i}{i+z}) \in \mathfrak{R}_N(i)$. This concludes that the class $\mathfrak{R}_N(i)$ is dense in $H^p(\mathbb{C}_{+1})$. The proof of Theorem 1 is complete.

The Corollary can be proved similarly.

Proof of Theorem 2 According to Lemma 1 and 2, there exist two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_{+1})$, $Q_k \in H^p(\mathbb{C}_{-1})$,

$$\sum_{k=1}^{\infty} \left(\|P_k\|_p^p + \|Q_k\|_p^p\right) \le 2\left(1 + \frac{2\pi}{1-p}\right)\|f\|_p^p$$

and

$$\lim_{n\to\infty}||f-\sum_{k=1}^{n}(P_k+Q_k)||_p=0.$$

Since

$$\|P_k\|^p_{H^p_{+1}}=\|P_k\|^p_p \ \text{ and } \ \|Q_k\|^p_{H^p_{-1}}=\|Q_k\|^p_p,$$

we have that (1) and (2) hold. For any $\delta>0, y>0$, the functions $|P|^p$ and $|Q|^p$ are subharmonic. Hence,

$$\left|\sum_{k=1}^{n}P_k(x+iy+i\delta)\right|^p\le\sum_{k=1}^{n}|P_k(x+iy+i\delta)|^p\le\frac{2}{\pi\delta}\sum_{k=1}^{n}\|P_k\|^p_p.$$

This implies that the series

$$\sum_{k=1}^{\infty}P_k(z)$$

uniformly converges in the closed upper half plane $\{z:\mathrm{Im}z\ge\delta\}$ for any $\delta>0$. As consequence, the function $g(z)$ is analytic in the upper half plane $\mathbb{C}_{+1}$. Similarly, we can prove that the function $h(z)$ is analytic in the lower half plane $\mathbb{C}_{-1}$. (1) implies that (3) holds. Therefore, the non-tangential boundary limits $g(x)$ and $h(x)$ of functions for $g\in H^p(\mathbb{C}_{+1})$ and $h\in H^p(\mathbb{C}_{-1})$ exist almost everywhere. (2) implies that $f(x)=g(x)+h(x)$ almost everywhere.

A new proof of Theorem B There exist $f(z)\in H^p(\mathbb{C}_{+1})$, $g(z)\in H^p(\mathbb{C}_{-1})$ such that $f(x)=g(x),\ \ a.e.x\in\mathbb{R}$. By Theorem 1 and Corollary 1, for any $\varepsilon>$, there exist $R\in\mathfrak{R}_N(i)$ and $R_2\in\mathfrak{R}_N(-i)$ such that

$$\|f-R_1\|_{H^p_{+1}}=\|f-R_1\|_p<\frac{\varepsilon}{4},\ \ \|g-R_2\|_{H^p_{-1}}=\|f-R_2\|_p<\frac{\varepsilon}{4}.$$

By the definition of $R\in\mathfrak{R}_N(i)$ and $R_2\in\mathfrak{R}_N(-i)$, there exist polynomials P_1 and P_2 such that

$$R_1(z)=P_1(\beta(z)+1)(\beta(z)+1)^{N+1},\ \ R_2(z)=P_2((\beta(z))^{-1}+1)((\beta(z))^{-1}+1)^{N+1},$$

where $\beta(z)=\frac{i-z}{i+z}$.

Let $m>\max\{degP_1,\ degP_2\}+N+1$, and define, for each $\varphi\in\mathbb{R}$,

$$R(z,\varphi)=R_1(z)-\frac{(\beta(z))^m(R_1(z)-R_2(z))}{(\beta(z))^m-e^{i\varphi}}.$$

Notice that $\beta(x)=e^{i\theta(x)}$, where $\theta(x)=\arg(i-x)-\arg(x+i)\in(-\pi,\pi)$ for $x\in\mathbb{R}$. By Fubini's theorem,

$$\begin{aligned}J&=\int_{-\pi}^{+\pi}\int_{-\infty}^{+\infty}|R(x,\varphi)-R_1(x)|^p\ dxd\varphi\\&=\int_{-\infty}^{+\infty}\int_{-\pi}^{+\pi}\frac{|R_1(x)-R_2(x)|^p}{|(1-e^{i\varphi-im\theta(x)}|^p}\ d\varphi dx.\end{aligned}$$

Observing

$$\int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1-e^{i\varphi - im\theta(x)}|^p} = \int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1-e^{i\varphi}|^p} = \int_{-\pi}^{\pi} \frac{d\varphi}{\sin^p \frac{\varphi}{2}} \leq 4\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\frac{2}{\pi}\varphi)^p} \leq \frac{2\pi}{1-p},$$

we obtain

$$J \leq \frac{2^{1-p}\pi}{1-p} \int_{-\infty}^{+\infty} |R_1(x) - R_2(x)|^p dx.$$

Therefore, there is a real number φ such that

$$\int_{-\infty}^{+\infty} |R(x,\varphi) - R_1(x)|^p \ dx \leq \frac{2\pi}{1-p}((\varepsilon/4)^p + (\varepsilon/4)^p).$$

Therefore, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} |R(x,\varphi) - f(x)|^p \ dx \\ &\leq \int_{-\infty}^{+\infty} |R(x,\varphi) - R_1(x)|^p \ dx + \int_{-\infty}^{+\infty} |R_1(x) - f(x)|^p \ dx \\ &\leq (\varepsilon/4)^p + \frac{4\pi}{1-p}(\varepsilon/4)^p. \end{aligned}$$

So, $R(z) = R(z,\varphi) \in L^p(\mathbb{R})$ is a rational function of z. There is a polynomial P_3 with $\deg P_3 = N + 1 + \deg P_1$ such that $R(z) = P_3(\beta(z) + 1)$. So the poles of R are contained in $\{x_k : \ k = 0, 1, ..., m+1\}$, where

$$x_k = \alpha(e^{\frac{i(\varphi+2k\pi)}{m}}) = \tan^2(\frac{(\varphi + 2k\pi)}{2^m}).$$

Thus, $R(z) \in X^p$.

Proof of Theorem 3. Recall that the Paley-Wiener Theorem asserts that $g \in H^2(\mathbb{C}_{+1})$ if and only if $\hat{g} \in L^2(\mathbb{R})$ with the support $\mathrm{supp}\hat{g} \subset [0,\infty)$, such that

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \hat{g}(t) e^{itz} dt \qquad (z \in \mathbb{C}_{+1}).$$

In the case there holds the equality

$$\int_0^{\infty} |\hat{g}(t)|^2 dt = \|g\|^2_{H^2_{+1}}.$$

Let $0 < p \leq 1, \ f \in H^p(\mathbb{C}_{+1})$. For $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$. Then $|f|^p$ is subharmonic, and, for $y > 0$,

$$|f_\delta(x+iy)| \leqslant C_p \|f\|_{H^p_{+1}} \delta^{-\frac{1}{p}},$$

where $C_p^p = \frac{2}{\pi}$. Therefore

$$\int_{-\infty}^{\infty} |f_\delta(x+iy)|^2 dx \leqslant \int_{-\infty}^{\infty} |f_\delta(x+iy)|^p |f_\delta(x+iy)|^{2-p} dx \leqslant C_p^{2-p} \|f\|^2_{H^p_{+1}} \delta^{1-\frac{2}{p}},$$

and

$$\int_{-\infty}^{\infty}|f_\delta(x+iy)|dx=\int_{-\infty}^{\infty}|f_\delta(x+iy)|^p|f_\delta(x+iy)|^{1-p}dx\leqslant C_p^{1-p}\|f\|_{H^p_{+1}}\delta^{1-\frac{1}{p}}.$$

Therefore, $\operatorname{supp}\hat{f}_\delta\subset[0,\infty)$, and

$$f_\delta(z)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\hat{f}_\delta(s)e^{itz}dt. \tag{10}$$

For $y>0$, $f_\delta(x+iy)=(P_y*f_\delta)(x)$, where

$$P_y(x)=\operatorname{Re}\left(\frac{i}{\pi z}\right)=\frac{y}{\pi(x^2+y^2)}$$

is the Poisson kernel of the upper plane $\mathbb{C}_+$. It is well known that $f_\delta\in L^2(\mathbb{R})$, $P_y\in L^1(\mathbb{R})$, $\hat{P}_y(s)=e^{-|s|y}$ for almost all $s\in\mathbb{R}$, and $\hat{f}_{\delta+y}(s)=\hat{f}_\delta(s)e^{-|s|y}$. So, for almost all $s\in\mathbb{R}$, $\hat{f}_{\delta+y}(s)e^{|s|(\delta+y)}=\hat{f}_\delta(s)e^{|s|\delta}$. Hence, the function $F(s)=\hat{f}_\delta(s)e^{|s|\delta}$ is independent of $\delta>0$, with $\operatorname{supp}F\subset[0,\infty)$, and

$$\int_{-\infty}^{\infty}|F(s)|^2e^{-2|s|\delta}dt=\int_{-\infty}^{\infty}|\hat{f}_\delta(x)|^2dx$$

$$=\int_{-\infty}^{\infty}|f_\delta(x)|^2dx\leqslant C_p^{2-p}\|f\|^2_{H^p_{+1}}\delta^{1-\frac{2}{p}}.$$

Therefore, for any $\delta>0$,

$$|F(s)|=|\hat{f}_\delta(s)|e^{s\delta}\leqslant\|f_\delta\|_1e^{s\delta}\leqslant C_p^{1-p}\|f\|_{H^p_{+1}}e^{s\delta}\delta^{-B_p},$$

where $B_p=\frac{1}{p}-1\geqslant 0$. Since

$$\inf\{|s|\delta-B_p\log\delta:\delta>0\}=B_p-B_p(\log B_p-\log|s|),$$

we have

$$|F(s)|\leqslant C_p^{1-p}\|f\|_{H^p_{+1}}B_p^{-B_p}e^{B_p}|s|^{B_p}.$$

Thus F is a slowly increasing continuous function F whose support is contained in $[0,\infty)$. Letting $\delta\to 0$ in (10), we see that (7) holds. F can also be regard as a tempered distribution defined through

$$(F,\hat{\varphi})=\int_{\mathbb{R}}F(x)\hat{\varphi}(x)dx$$

for φ in the Schwarz class $\mathbb{S}$. So,

$$\lim_{\delta\to 0}\int_{\mathbb{R}}f_\delta(x)\varphi(x)dx=\lim_{\delta\to 0}\int_0^{+\infty}\hat{f}_\delta(x)\hat{\varphi}(x)dx$$

$$=\lim_{\delta\to 0}\int_0^{+\infty}F(x)e^{-\delta x}\hat{\varphi}(x)dx=(F,\hat{\varphi}).$$

This completes the proof of Theorem 3.

A proof of Theorem C. Let $1 \le p \le 2$. If $f \in L^p$ and $\text{supp}\hat{f} \subset [0,\infty)$, then

$$|\chi_{[0,\infty)}(t)e^{2\pi iz\cdot t}\hat{f}(t)| = \chi_{[0,\infty)}(t)|\hat{f}(t)|e^{-2\pi y\cdot t} \in L^1(\mathbb{R}^n),$$

where $\chi_{[0,\infty)}(t)$ is the characteristic function of $[0,\infty)$, that is, $\chi_{[0,\infty)}(t) = 1$, for $t \in [0,\infty)$, and otherwise zero. It is evident that the function

$$G(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{izt}\hat{f}(t)dt = \int_{\mathbb{R}} \chi_{[0,\infty)}(t)e^{izt}\hat{f}(t)dt$$

is holomorphic in $\mathbb{C}_{+1}$. To complete the proof of Theorem C, it is sufficient to prove that $G(z) \in H^p(\mathbb{C}_{+1})$ and the boundary limit of $G(z)$ is $f(x)$ as $y \to 0$. Fix $z \in \mathbb{C}_{+1}$ and let

$$g_z(t) = \chi_{[0,\infty)}(t)\frac{e^{izt}}{\sqrt{2\pi}}, \quad \widetilde{g}_z(t) = g_z(-t)$$

then $g_z \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, $\hat{g}_z(s) = \frac{1}{2\pi i(s-z)}$ and

$$\begin{aligned} G(z) = & \tfrac{1}{\sqrt{2\pi}}\textstyle\int_{\mathbb{R}}\chi_{[0,\infty)}(t)e^{izt}\tfrac{1}{\sqrt{2\pi}}(\int_{\mathbb{R}} e^{-ist}F(s)ds)dt \\ = & \tfrac{1}{2\pi i}\textstyle\int_{\mathbb{R}}\tfrac{f(s)ds}{s-z}. \end{aligned}.$$

For $z, w \in \mathbb{C}_{+1}$, let

$$I(z,w) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{f(t)dt}{(t-z)(t-w)}.$$

Then

$$I(z,w) = \int_{\mathbb{R}}\hat{g}_z(t)f(t)\hat{g}_w(-t)dt.$$

For $z, w \in \mathbb{C}_{+1}$, $\sqrt{2\pi}\hat{g}_z(t)\hat{g}_w(-t) = \widehat{g_z * \widetilde{g}_w}(t)$, where

$$(g_z * \widetilde{g}_w)(t) = \int_{\mathbb{R}} g_z(\xi)\widetilde{g}_w(t-\xi)d\xi = \int_{\mathbb{R}} g_z(\xi)g_w(\xi - t)d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\chi_{[0,\infty)}(\xi)e^{2\pi iz\xi}\chi_{[0,\infty)}(\xi - t)e^{2\pi iw(\xi-t)}d\xi.$$

Therefore,

$$I(z,w) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(s)\chi_{[0,\infty)}(s)(g_z * \widetilde{g}_w)(s)ds$$

$$= \frac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}}\hat{f}(s)\chi_{[0,\infty)}(s)\int_{\mathbb{R}}\chi_{[0,\infty)}(\xi)e^{2\pi iz\cdot\xi}\chi_{[0,\infty)}(\xi - s)e^{2\pi iw\cdot(\xi-s)}d\xi ds.$$

By Fubini's theorem and the relation

$$\chi_{[0,\infty)}(t)\chi_{[0,\infty)}(t+s)\chi_{[0,\infty)}(s) = \chi_{[0,\infty)}(t)\chi_{[0,\infty)}(s),$$

we have

$$\begin{aligned} I(z,w) = & \frac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{[0,\infty)}(s)\chi_{[0,\infty)}(t)\chi_{[0,\infty)}(t+s)e^{iz(s+t)}e^{iwt}\hat{f}(s)dsdt \\ = & \frac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{[0,\infty)}(t)\chi_{[0,\infty)}(s)e^{izs}e^{i(z+w)t}\hat{f}(s)dtds \\ = & \frac{i}{2\pi}\frac{G(z)}{z+w}. \end{aligned}$$

Thus, for $z\in\mathbb{C}_{+1}$, we have $-\bar{z}\in\mathbb{C}_{+1}$, and

$$I(z,-\bar{z}) = \frac{i}{2\pi}\frac{G(z)}{z-\bar{z}} = \frac{G(z)}{4\pi y},\ z=x+iy, y>0.$$

So,

$$G(z) = \int_{\mathbb{R}}\frac{4\pi y f(t)dt}{(2\pi)^2(t-z)(t-\bar{z})} = \int_{\mathbb{R}} f(t)P(x-t,y)dt,$$

where $P(x,y)=\frac{y}{\pi(x^2+y^2)}$ is the Poisson Kernel of the upper half plane $\mathbb{C}_{+1}$. Therefore, the boundary limit of $G(z)$ is $f(x)$ as $y\to 0$ and $G(z)\in H^p(\mathbb{C}_{+1})$. The proof of Theorem C is complete.

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